PROM^2 for Girls 2025
Team Round Solutions

PRISMS Math Team

April 2025

Problem 1
Every year, PROM^2 for Girls provides free T-shirts for participants (big thanks to our sponsors!). We always choose Custom Ink to produce these T-shirts. The unit price varies based on the number of T-shirts people place in order. The price table is attached below.

Number of T-shirts	1-20	21-50	51-100	101-200	201-500	>500
Unit Price	\$14.5	\$13.5	\$12.5	\$11.5	\$10.5	\$9.5

Last year, we ordered 135 T-shirts at the beginning. Later, we found that more students registered, so we place a 2nd order of 80 T-shirts. How much money would we save if we combine those two orders as one at the beginning?

Answer: 295
Solution:
$11.5\times135+12.5\times80-10.5\times(135+80) = 135\times(11.5-10.5)+80\times(12.5-10.5) = 135 + 80 \times 2 = \boxed{295}$.
Credit: Joseph Li

Problem 2
Sophia qualified for MOP last year because she did well in USAMO, which is a test that has 6 questions. For each question, a participant could get 0, 1, 2, 3, 4, 5, 6, or 7 credits. If the cutoff for qualifying for the next level, MOP, is 37, what is the least amount of questions Sophia needed to get full credit to qualify for MOP?

Answer: 1
Solution:
If Sophia didn't get full credit for any questions, she would get at most $6 \cdot 6 = 36 < 37$ credits, which isn't enough! Thus, Sophia must have received full credit

for at least 1 question. In fact, by trying to have as many 6's as possible, we get that Sophia could get 37 if she got full credit for only $\boxed{1}$ question, such as by having $7+6+6+6+6+6=37$.
Credit: Charlie Gu

Problem 3
PRISMS students all love ice cream! For a party, Elisa, the MC of PROM^2, wants to get 3 gallons of ice cream for all the volunteers. The ice creams come in three sizes: pints, quarts, and gallons. If there are four quarts in a gallon and two pints in a quart, how many ways can she choose the sizes of the ice cream to get 3 gallons of ice cream?

Answer: 28
Solution:
Let's count it by cases.

- Case 1: If Elisa chooses 3 one-gallon-sized ice cream, she doesn't need anything else.
- Case 2: If Elisa chooses 2 one-gallon-sized ice cream, she needs some quarts and pints to fill another gallon of ice cream. The number of pints is determined by the number of quarts, and she can choose 0-4 quarts. Thus, she has 5 ways.
- Case 3: If Elisa chooses 1 one-gallon-sized ice cream, she needs some quarts and pints to fill 2 gallons of ice cream. Similarly, Elisa can choose 0-8 quarts, so she has 9 ways.
- Case 4: If Elisa chooses 0 one-gallon-sized ice cream, she needs some quarts and pints to fill 3 gallons of ice cream. Similarly, Elisa can choose 0-12 quarts, so she has 13 ways.

Summing our cases together, the desired answer is $1+5+9+13=\boxed{28}$.
Credit: Sophia Zhang

Problem 4
Find all the solutions to the equation

$$\frac{\max(21,x)+\min(9,x)}{\max(9,x)-\min(3,x)}=\frac{17}{13},$$

where $\max(a,b)$ is denoted as the larger number between a and b, and $\min(a,b)$ as the smaller number.

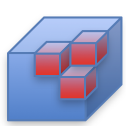
Answer: 42, -4
Solution:
We consider different intervals for x.

- When $x \geq 21$, the equation becomes $\frac{x+9}{x-3} = \frac{17}{13}$, so $x = 42$.
- When $21 > x \geq 9$, the equation becomes $\frac{21+9}{x-3} = \frac{17}{13}$. This gives $x = 25\frac{16}{17}$ but since $25\frac{16}{17} > 21$, there's no solution to this case.
- When $9 > x \geq 3$, the equation becomes $\frac{21+x}{9-3} = \frac{17}{13}$. This gives $x = -13\frac{2}{17}$ but since $-13\frac{2}{17} < 3$, there's also no solution to this case.
- When $x < 3$, the equation becomes $\frac{21+x}{9-x} = \frac{17}{13}$, so $x = -4$, which is in the interval of this case.

Finishing, our solutions are $x = \boxed{42, -4}$.
Credits: Kris Sheng, Joseph Li

Problem 5
Twenty-seven unit cubes are glued together to form a larger cube. Some of the faces of the large cube are painted red and others faces are painted blue. Denote R as the number of unit cubes that at least one face is painted red, and B as the number of unit cubes that at least one face is painted blue. Find the maximum possible value of $R + B$.

Answer: 42
Solution:
When we color the top face and bottom face red, and all other four faces blue, $R = 18$ and $B = 24$, so $R + B = 42$.
Next, we will prove that we must have $R + B \leq 42$. Let x be the number of unit cubes that has at least one red face and at least one blue face (let's call them colorful cubes). Because the unit cube at the center of the large cube can not be painted, $R + B \leq (27 - 1) + x = 26 + x$. Now let's find out the maximum possible value for x.
For any colorful unit cube, it has at least two faces on the surface of the larger cube. These cubes are the 8 unit cubes at the vertices and the 12 unit cubes at the middle of edges.

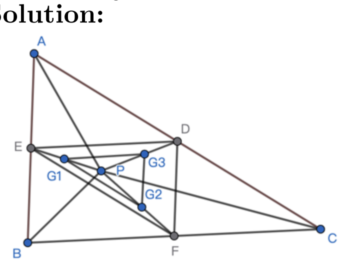


For the 12 unit cubes at the middle of edges, they can be assigned into 4 groups such that the 3 unit cubes are on the edges connecting to one vertex. For those 3 unit cubes in the same group, (like the 3 unit cubes shown above), their colors

are determined by the colors used for the top face, the right face, and the front face of the large cube. By the Pigeonhole Principle, at least 2 of those 3 faces are painted by the same color. For instance, the top face and the front face are both painted red, then the unit cube at the middle of the edge between those 2 faces can't be painted by two colors. That said, for any 3 unit cubes in the same group, at most 2 of them are colorful cubes. Thus, among all 12 unit cubes at the middle of edges, at most $4 \cdot 2 = 8$ of them are colorful cubes. So $x \leq 8 + 8 = 16$.
Finally, $R + B \leq 26 + x \leq 26 + 16 = \boxed{42}$.
Credit: Sigma Liu

Problem 6
Jessie draws a triangle ABC with side lengths 3, 4, 5. She finds that the three angle bisectors are concurrent at a point P in the triangle. Let G_1, G_2, G_3 be the centroids of the triangles PAB, PAC, and PBC, respectively. What is the area of the triangle $G_1G_2G_3$?
(The intersection of three angle bisectors of a triangle is the incenter of the triangle, and the intersection of three medians of a triangle is the centroid of the triangle.)

Answer: $\frac{2}{3}$
Solution:



Let D, E, F be the midpoints of AC, AB, and BC, respectively. The property of centroid tells us that G_1 is on median PE, $PG_1 = \frac{2}{3}PE$. Similarly, $PG_2 = \frac{2}{3}PF$ and $PG_3 = \frac{2}{3}PD$.
Then, we know triangle $G_1G_2G_3$ is similar to triangle DEF, with the ratio is $2:3$. At the same time, as D, E, F are the middle points of AC, AB, and BC, triangle DEF is similar to triangle ABC, with ratio is $1:2$. Because $3^2 + 4^2 = 5^2$, Pythagorean theorem gives triangle ABC is a right triangle. So its area is $\frac{3 \cdot 4}{2} = 6$.
Using properties of similar triangles, the area of triangle DEF is $6 \cdot (\frac{1}{2})^2 = \frac{3}{2}$.
Finally, the area of triangle $G_1G_2G_3$ is $\frac{3}{2} \cdot (\frac{2}{3})^2 = \boxed{\frac{2}{3}}$.
Credit: Terry Huang

Problem 7
Catherine colored 30 points red in the coordinates plane, where none three of those red points are colinear. She connected some pairs of red points by line segments. For each red point, define the degree of this point as the number of line segments connecting to it. Catherine labelled each red point with the square of its degree, and labelled each line segment with the sum of the degrees of its two endpoints. She subtracted the sum of numbers at all red points by the sum of numbers at all line segments. What is the maximum possible results Catherine could get?

Answer: 0
Solution:
Suppose the degrees of those points are $d_1, d_2, ..., d_{30}$. The sum of numbers at all red points is $d_1^2 + d_2^2 + ... + d_{30}^2$.
The number at each line segment is in the form $d_i + d_j$, where $1 \leq i < j \leq 30$. For each $1 \leq k \leq 30$, the value d_k is added at each line segment that connect to the kth point. As its degree is d_k, there are d_k line segments connecting to it. Thus d_k is added d_k times, which means its total contribution to the sum is d_k^2. Therefore, the total sum of all $d_i + d_j$ is also $d_1^2 + d_2^2 + ... + d_{30}^2$, so, the desired difference is always $\boxed{0}$.
Credit: Sophia Zhang

Problem 8
Your origami instructors, Jessie and Melinda, start to read the book ¡The Book Thief¿ on the same day. Melinda reads 16 pages every day. Jessie reads 5 pages on the first day. From the 2nd day to the 15th day, each day she reads 2 more pages than she does on the previous day (thus, she reads 33 pages on the 15th day). Since the 16th day, she starts to prepare for the Origami Tournament and each day she reads 2 less pages than she does on the previous day. On the last day, she reads 1 page and finishes the entire book. By the end of which day, the number of pages that Jessie has read more than Melinda has is maximized?

Answer: 23
Solution:
Denote P_n as the number of pages that Jessie has read more than Melinda has after the nth day. We just need to realize the following facts:
If on the nth day, Jessie reads more than Melinda does, then $P_n > P_{n-1}$, as $P_n - P_{n-1}$ is the number of pages that Jessie reads more than Melinda does on that day.
Similarly, if on the nth day, Jessie reads less than Melinda does, then $P_n < P_{n-1}$. On the 23rd day, Jessie reads $33 - (23 - 15) \cdot 2 = 17$ pages, and on the previous

days, she reads even more. Thus, ... $< P_{20} < P_{21} < P_{22} < P_{23}$.
On the 24th day, Jessie reads 15 pages, and she reads less on the following days. Thus $P_{23} > P_{24} > P_{25} > P_{26} > ....$ Looking across all P_n, we can see that P_{23} is the maximum number so this occurs on day $\boxed{23}$.
Credit: Joseph Li

Problem 9
Find all the positive integers $a < b$ such that $a^3 + (a+1)^3 + (a+2)^3 + ... + b^3 = 12345678$.

Answer: no integer solutions
Solution:
From the formula $1^3 + 2^3 + 3^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$, we have $a^3 + (a+1)^3 + (a+2)^3 + ... + b^3 = (1^3 + 2^3 + ... + b^3) - (1^3 + 2^3 + ... + (a-1)^3) = (\frac{b(b+1)}{2})^2 - (\frac{(a-1)a}{2})^2$.
Let $x = \frac{b(b+1)}{2}$, $y = \frac{(a-1)a}{2}$. Then x, y are integers and $x^2 - y^2 = 12345678$.
If x, y have different parity, then $x^2 - y^2$ is odd and can't be 12345678.
If x, y have the same parity, then $x^2 - y^2$ is a multiple of 4 and also can't be 12345678. Thus, so there are $\boxed{\text{no integer solutions}}$.
Credit: Sophia Zhang

Problem 10
Let $n > 3$ be a positive integer. The sum of the measures of some (not all) interior angles of a regular polygon with n sides is $3300°$. Find all the possible values of n.

Answer: 24, 57
Solution:
Suppose $3300°$ is the sum of k interior angles. Then, $k < n$ and $\frac{(n-2)\cdot 180°}{n} \cdot k = 3300° \implies 3k(n-2) = 55n$.
If n is odd, then n and $n-2$ are co-prime. Thus, $n|3k$. As $k < n$, $3k < 3n$, so $3k = n$ or $2n$. Because $55n$ is odd, $3k$ has to be odd. Therefore, $3k = n$. This implies $n - 2 = 55$ and $n = 57$.
If n is even, suppose $n = 2m \implies 3k(m-1) = 55m$. Thus, m must be a multiple of 3. Suppose $m = 3t$, then $k(3t-1) = 55t$.
Because t and $3t - 1$ are co-prime. Thus, $t|k$. As $k < n = 6t$, so $k = at$, where $1 \leq a \leq 5$. Then $a(3t-1) = 55$, which is $3at = 55 + a$. As $55 + a$ is a multiple of 3 and $1 \leq a \leq 5$, $a = 2, 5$.
If $a = 2$ the equation becomes $6t = 57$, which has no integer root.
If $a = 5$ then $t = 4$, and $n = 6t = 24$.
So there are two solutions of $n = \boxed{24, 57}$.
Credit: Joseph Li