PROM^2 For Girls 2024 Solutions

PRISMS Math Team

April 2024

Answer key: DCACD BCCBD CDBAC DEDDC DBEDC BBCCB

1. (Peter) Evaluate $(2+0+2+4)\cdot(2^2+0^2+2^2+4^2)$.

   (A) 64 (B) 128 (C) 160 (D) 192 (E) 240

   Answer: 192

   Solution:

$$\begin{aligned}(2+0+2+4)(2^2+0^2+2^2+4^2) &= (2+0+2+4)(4+0+4+16)\\ &= 8\cdot 24 = \boxed{\text{(D) } 192}.\end{aligned}$$

2. (Kevin Yang) The student government had a bake sale every day from April 15th to April 21st. Every day, they sold at most 15 cupcakes. They end up selling 100 cupcakes by the end of the week. What is the least number of cupcakes they sold on any day?

   (A) 5 (B) 8 (C) 10 (D) 12 (E) 14

   Answer: 10

   Solution: Assume they sell as many cupcakes as they can on the other six days, then the least number of cupcakes is $100-15\cdot 6 = \boxed{\text{(B) } 10}$.

3. (Asel) Let N be the smallest positive integer that gives the following remainders: 1 when divided by 2, 2 when divided by 3, 3 when divided by 4, 4 when divided by 5, 5 when divided by 6. Which of the following intervals does N lie in?

   (A) $[50,60)$ (B) $[70,80)$ (C) $[110,120)$ (D) $[120,130)$ (E) $[710,720)$

   Answer: [50, 60)

   Solution: $N+1$ gives remainders $0 \pmod{2,3,4,5,6}$. Therefore, $N+1 = lcm(2,3,4,5,6) = 60$, and $N = 59$, which lies in the interval $\boxed{\text{(A) } [50, 60)}$.

4. (Mr. Li) Evaluate

$$\frac{2024-20.24}{33}.$$

   (A) 11.18 (B) 20.24 (C) 60.72 (D) 62 (E) 66

   Answer: 60.72

Solution:

$$\begin{aligned}\frac{2024-20.24}{33} &= \frac{20.24\cdot 100-20.24}{33}\\ &= \frac{20.24\cdot 99}{33} = 20.24\cdot 3\\ &= \boxed{\text{(C) } 60.72}.\end{aligned}$$

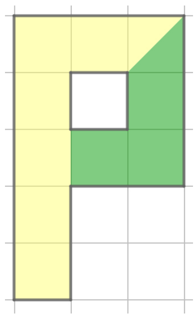
5. (Rafi) Given that the 3-digit integer $\overline{3A7}$ is multiple of 9, find the value of A. (For example, if $A = 0$, $\overline{3A7} = 307$.)

   (A) 1 (B) 2 (C) 5 (D) 8 (E) 9

   Answer: 8

   Solution: $\overline{3A7} = 307 + 10A \equiv 1 + A \pmod 9$. Therefore, $A = \boxed{\text{(D) } 8}$.

6. (Peter) As shown below, the letter P has been divided into two regions. Given that each square grid has side length 1, what is the difference between the areas of the yellow region and the green region?

   (A) 2 (B) 3 (C) 3.5 (D) 4 (E) 6.5

   Answer: 3

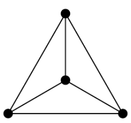
   Solution: Area of each square grid is $1 \times 1 = 1$. Since yellow region has 6.5 grids and green region has 3.5 grids, the area difference is $1 \cdot (6.5 - 3.5) = \boxed{\text{(B) } 3}$.

7. (Jessie) There are n points in the plane. Every two of them are connected with a line segment. What is the maximum value of n such that all of the segments do not intersect each other except for at the vertices?

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

   Answer: 4

   Solution: When $n = 4$, we have the following way to connect 4 points:

   When $n = 5$, there are 5 points, 10 faces, and 10 segments. $|V| - |E| + |F| = 5 > 2$. By Euler's Formula, some segments must intersect each other.

Therefore, maximum $n = \boxed{\text{(C) } 4}$.

8. (Eva) 7 students, Asel, Ben, Cedric, David, Eva, Felicity, Grace sit around a round table clockwise in that order. Asel calls out 1, Ben calls out 2, Cedric calls out 3, and the rest continue repeating calling out 1, 2, then 3. When someone says 3, they will leave the table immediately and the counting continues clockwise. After they continue this process for a while, there is only one person still sitting at the table. Who is it?

   (A) Asel (B) Ben (C) David (D) Eva (E) Grace

   Answer: David

   Solution: We draw the table with the people, and cross them out as we count.

1	A	D	G	D	A	A
2	B	E	A	E	D	D
3	C	F	B	G	E	A

   Therefore, the only one remaining is $\boxed{\text{(C) David}}$.

9. (Josh) An arithmetic sequence is a sequence of numbers such that the difference from any succeeding term to its preceding term remains constant throughout the sequence. An arithmetic sequence has its first four terms $a, 2a, b, 6-b-a$. Find its 100th term.

   (A) $\frac{100}{3}$ (B) 75 (C) 100 (D) $\frac{400}{3}$ (E) 300

   Answer: 75

   Solution: By definition of arithmetic sequence, $2a - a = b - 2a = 6 - b - a$. Therefore, $b = 3a$ and $4a = 6 - b - a = 6 - 4a$, which gives $a = \frac{6}{8} = \frac{3}{4}$.

   Therefore, its 100th term is $100a = \frac{3}{4} \cdot 100 = \boxed{\text{(D) } 75}$.

10. (Sophia) Wendy is shooting an arrow on a board with a rectangular target of size 14in × 34in. Given that her arrow lands randomly on any point on the board with equal chance, and the probability the arrow hits the target is 17%, what is the total area of the board not inside the target (in in^2)?

    (A) 476 (B) 924 (C) 1400 (D) 2324 (E) 2800

    Answer: 2324

    Solution: Area of the board not inside the target is

$$\begin{aligned} 14 \cdot 34 \cdot \frac{1}{17\%} - 14 \cdot 34 &= 14 \cdot 34 \cdot \frac{100}{17} - 14 \cdot 34 \\ &= 2800 - 476 = \boxed{\text{(D) } 2324}. \end{aligned}$$

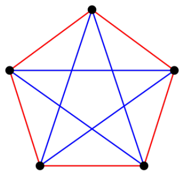
11. (Maggie) The math team is holding a party. There are n participants, and every two of them are either friends or strangers, What is the minimum possible number of participants, such that there must be at least one group of three people, such that either every two of them are

strangers, or every two of them are friends?

(A) 4 (B) 5 (C) 6 (D) 8 (E) 9

Answer: 6

Solution: When there are 5 participants, in the following relationship (red line means they're friends, and blue line means they're strangers), we can't find a group of 3 satisfying the condition.



However, when there are 6 participants A, B, C, D, E, F, for any participant A, by Pigeonhole's Principle, they have either ≥ 3 friends or ≥ 3 strangers. Without loss of generality, assume A is friends with B, C, D.

If B, C, D are all strangers with each other, then they form a group of 3 satisfying hte condition. Otherwise, if any of them are friends (assume B and C), then A, B, C are a group of 3 satisfying the condition.

Therefore, the minimum number of participants ensuring such a group of 3 exists is $\boxed{\text{(C) 6.}}$

12. (Felicity) Felicity was taking Honors Biology. The final grade consists of 40% semester 1 score, 40% semester 2 score, and 20% project score. A grade greater than or equal to 93 is an A. If she got 88 in semester 1, what's the minimum score she needs in semester 2 to get an A in final grade? (The possible scores are integers from 0 to 100)

    (A) 92 (B) 93 (C) 94 (D) 95 (E) 98

    Answer: 95

    Solution: Assume her project score is as high as possible, and let her semester 2 score be x. Her final score is $40\% \cdot 88 + 40\% \cdot x + 20\% \cdot 100 = 0.4x + 55.2 \geq 93$.

    Therefore, $0.4x \geq 37.8$. $x \geq \frac{37.8}{0.4} = 94.5$.

    Since possible scores are integers, the minimum score she needs in semester 2 is $\boxed{\text{(D) 95}}$.

13. (Peter) While the $PROM^2$ Academy is preparing lunch, the staff realize that the number of eggs is one less than a multiple of 12, as indicated by the dozen boxes. However, when they arrange the eggs into rows of 7, there is none left. What interval does the minimum number of eggs possible lie?

    (A) $[20, 30)$ (B) $[30, 40)$ (C) $[40, 50)$ (D) $[50, 60)$ (E) $[60, 70)$

    Answer: [30, 40)

    Solution: We list the numbers that are one less than a multiple of 12. $11 \equiv 4 \pmod 7$, $23 \equiv 2 \pmod 7$, and $35 \equiv 0 \pmod 7$. Therefore, the minimum number of eggs, 35, lies in

(B) [30, 40).

14. (Spencer) There are 100 towns in $PROM^2$ City, and 98 roads are built among them. Each road connects two towns, and there is at most one road between each pair of towns. A "group" is a set of towns, such that two towns are in the same group if and only if one can go from one town to another through one or more roads. What is the minimum possible number of groups in $PROM^2$ City?

    (A) 2 (B) 3 (C) 50 (D) 98 (E) 99

    Answer: 2

    Solution: When no roads were built, there are 100 groups. Each new road connects two towns.

    (a) Two towns were in the same group: The new road doesn't change number of groups.

    (b) Two towns were in different groups: The new road merges these two groups, and the number of groups decreases by 1.

    This means number of groups after building 98 roads $\geq 100 - 98 = 2$. We can build 98 roads between town 1 and town 2, town 2 and town 3, ..., town 98 and town 99. Then there are two groups: town 1 to town 99, and town 100. Therefore, the minimum number of groups (A) 2.

15. (Sophia) Annie has a bunny named Pumpkin. Annie's friends, Amy, Olivia, and Sophia, all adore Pumpkin and like to bring carrots and lettuce for Pumpkin every few days. Amy visits once every 4 days, Olivia visits once every 5 days, Sophia visits every Wednesday and Saturday of the week. Pumpkin's birthday is on January 1st, a Wednesday, and Amy, Olivia, and Sophia all visit on that day. In the month of January, how many days will at least one of Annie's friends come visit Pumpkin? (January has 31 days).

    (A) 5 (B) 17 (C) 18 (D) 19 (E) 23

    Answer: 18

    Solution: We draw the calendar and simulate who comes on which day.

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

Therefore, there are (C) 18 days when at least one of Annie's friends visit Pumpkin.

16. (Peter) In the card game *Egyptian Rat Screw*, the pile of played cards can be slapped when either: (1) the top 2 cards are of the same rank, or (2) the top card is of the same rank as the third from the top. Right now the pile is empty, and David keeps adding cards to the top of the pile. Given that each card has an integer rank between 2 and 10, inclusive, how many

different combinations of ranks are there such that when the third card is added, the pile can be slapped for the first time?

(A) 72 (B) 81 (C) 90 (D) 144 (E) 162

Answer: 144

Solution: Since the pile is slapped for the first time after adding the third card, the first two cards are of different ranks, which gives $9 \cdot 8 = 72$ combinations.

The third card is of the same rank as either the first or the second card, which means there are $72 \cdot 2 = \boxed{\text{(D) 144}}$ different combinations.

17. (Peter) There are four townships in $PROM^2$ City, named Alpha, Bravo, Charlie, and Delta. Given that Alpha is 12 miles away from Bravo, Bravo is 8 miles away from Charlie, and Charlie is 6 miles away from Delta. Which of the following is not a possible distance, in miles, between Alpha and Delta?

(A) $\frac{\sqrt{2}}{\pi^\pi}$ (B) 6 (C) 12 (D) 20 (E) 30

Answer: 30

Solution: The shortest distance between two points is the straight line joining them. Therefore, the distance between Alpha and Delta is less than or equal to $12 + 8 + 6 = 26$. Since $30 > 26$, $\boxed{\text{(D) 30}}$ isn't a possible distance.

18. (Mr. Li) Denote $\sigma(n)$ as the sum of all the divisors of n. For example, $\sigma(12) = 1+2+3+4+6+12 = 28$. Evaluate $\sigma(960) - \sigma(480)$.

(A) 480 (B) 960 (C) 1440 (D) 1536 (E) 1920

Answer: 1536

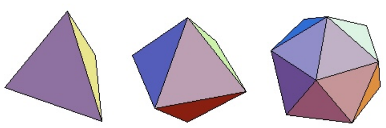
Solution: $\sigma(960) - \sigma(480)$ is the sum of divisors of 960 that don't divide 480.

$960 = 2^6 \cdot 3 \cdot 5$, and $480 = 2^5 \cdot 3 \cdot 5$. Let a divisor of 960 be $d = 2^a \cdot 3^b \cdot 5^c$ where $a \leq 6, b \leq 1$, and $c \leq 1$. Since $d \nmid 480$, either $a > 5$, $b > 1$, or $c > 1$.

Combining these two conditions, since $b \leq 1$ and $c \leq 1$, we have $a = 6$ and $d = 2^6 \cdot 3^b \cdot 5^c$.

$\sigma(960) - \sigma(480)$ is sum of all such d's, which equals $2^6 \cdot 3^0 \cdot 5^0 + 2^6 \cdot 3^1 \cdot 5^0 + 2^6 \cdot 3^0 \cdot 5^1 + 2^6 \cdot 3^1 \cdot 5^1 = 64 \cdot (1+3) \cdot (1+5) = \boxed{\text{(D) 1536}}$.

19. (Jessie) Alice has 52 equilateral triangles. She could only stick 4 together to form a tetrahedron (4 vertices), or 8 together to form an octahedron (6 vertices), or 20 together to form an icosahedron (12 vertices). Given that she needs to use all of the triangles she had, what is the minimum number of the sum of the number of vertices?



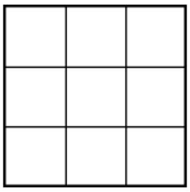
(A) 22 (B) 30 (C) 34 (D) 40 (E) 52

Answer: 34

Solution: For a tetrahedron, each triangle contributes to $\frac{4}{4} = 1$ vertex. For an octahedron, each triangle contributes to $\frac{6}{8} = \frac{3}{4}$ vertices. For an icosahedron, each triangle contributes to $\frac{12}{20} = \frac{3}{5}$ vertices. Therefore, we want as many icosahedrons as possible, so each individual triangle contributes to a minimum number of vertices.

Since $52 = 20 \cdot 2 + 8 + 4 < 60 = 3 \cdot 20$, we can have 2 icosahedrons, 1 octahedron, and 1 tetrahedron, which have $2 \cdot 12 + 6 + 4 = \boxed{\text{(C) } 34}$ vertices.

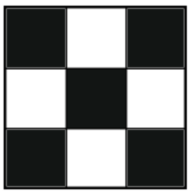
20. (David) David has to mop the floor with 9 square cells as shown below. He can begin mopping in any cell and then move to an adjacent cell in every turn. (Two cells are adjacent if and only if they share an edge.) In how many ways can David mop the entire floor without revisiting any cell?



(A) 8 (B) 24 (C) 40 (D) 48 (E) 64

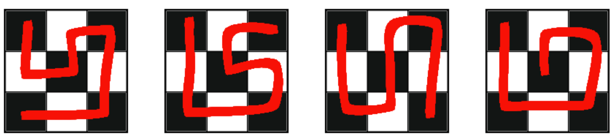
Answer: 40

Solution: We color the floor as shown below.



In every move, he can only go from a black cell to a white cell, or vice versa. To mop all 5 black cells and 4 white cells without revisiting any cell, he must start from a black cell. Therefore, we have two possible cases.

- David starts at a corner: Without loss of generality, assume he starts from the up left corner. He can either move right or down. Without loss of generality, assume he first moves down. There are 4 ways, as shown below.

  Hence, by symmetry, there are $4 \cdot 2 \cdot 4 = 32$ ways.
- David starts at the center: After moving to one of the 4 edge cells, he can only go around either clockwise or counterclockwise, which are $4 \cdot 2 = 8$ ways.

Hence, David has $32 + 8 = \boxed{\text{(C) } 40}$ ways to mop the floor.

21. (Sigma Liu) Find the greatest radius of a circle inscribed in a triangle inscribed in the unit circle.

(A) $\frac{1}{3}$ (B) $\sqrt{2}-1$ (C) $\frac{\sqrt{3}}{3}$ (D) $\frac{1}{2}$ (E) $\frac{\sqrt{6}}{2}$

Answer: $\frac{1}{2}$

Solution: Let the three internal angles of the triangle be α, β, and $\gamma = \pi - \alpha - \beta$. Then the three side lengths are respectively $2\sin\alpha, 2\sin\beta$, and $2\sin(\alpha+\beta)$. The radius of its inscribed circle is area divided by semi-perimeter, which equals

$$\begin{aligned} R &= \frac{1}{2} \cdot \frac{\sin(2\alpha) + \sin(2\beta) - \sin(2\alpha + 2\beta)}{\sin\alpha + \sin\beta + \sin(\alpha+\beta)} \\ &= \frac{\sin(\alpha+\beta)\cos(\alpha-\beta) - \sin(\alpha+\beta)\cos(\alpha+\beta)}{\sin\alpha + \sin\beta + \sin(\alpha+\beta)} \\ &= \frac{2\sin\alpha\sin\beta\sin(\alpha+\beta)}{\sin\alpha + \sin\beta + \sin(\alpha+\beta)} \end{aligned}$$

Since $\sin\alpha$, $\sin\beta$, and $\sin\gamma$ are all positive, we can apply the AM-GM inequality here.

$$\begin{aligned} R &\le \frac{2\sin\alpha\sin\beta\sin\gamma}{3(\sin\alpha\sin\beta\sin\gamma)^{1/3}} \\ &= \frac{2}{3}(\sin\alpha\sin\beta\sin\gamma)^{2/3} \\ &\le \frac{2}{3}\left(\frac{\sin\alpha + \sin\beta + \sin\gamma}{3}\right)^2 \\ &\le \frac{2}{3}(\sin(\frac{\pi}{3}))^2 \\ &= \boxed{\text{(C)}\tfrac{1}{2}}. \end{aligned}$$

Remark: One can easily observe that the maximum radius is attained when the triangle is equilateral. Thus the maximum radius is simply given by $R = \sin(\frac{\pi}{6}) = \frac{1}{2}$.

22. (Sophia) Find the number of ways to tile a 3×10 board with 3×1 or 1×3 tiles.

(A) 24 (B) 28 (C) 32 (D) 40 (E) 42

Answer: 28

Solution: For each column, there are four possible states as shown below.

State 2 must follow state 1, and state 3 must follow state 2. State 1 and state 4 can follow either state 3 or state 4. Therefore, we recursively list the number of ways to tile a $3 \times n$ board when the last column is in each state.

n	1	2	3	4	5	6	7	8	9	10
1	1	1	1	2	3	4	6	9	13	19
2	0	1	1	1	2	3	4	6	9	13
3	0	0	1	1	1	2	3	4	6	9
4	1	1	1	2	3	4	6	9	13	19

The last row can be in either state 3 or 4. There are in total $19 + 9 = \boxed{\text{(B) } 28}$ ways.

23. (Kevin Yang) Integers x, y, z, w satisfy $3^{x+2} - 3^{y+0} + 3^{z+2} - 3^{w+4} = 2024$. Find $x + y + z + w$.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Answer: 8

Solution: Observe that $2024 = 2025 - 1$, and $2025 = 45^2$. Therefore,

$$\begin{aligned} 2024 &= 45^2 - 1 = (3^2 \cdot 5)^2 - 1 \\ &= 3^4 \cdot 25 - 1 = 3^4 \cdot (3^3 - 2) - 1 \\ &= 3^7 - 2 \cdot 3^4 - 3^0 \\ &= 3^7 + 3^5 - 3^4 - 3^0. \end{aligned}$$

Therefore, $x + y + z + w = 7 + 5 + 4 + 0 - 2 - 0 - 2 - 4 = \boxed{\text{(E) } 8}$.

Alternatively, in ternary expansion, $(2024)_{10} = (2202222)_3$, and we can get the same answer.

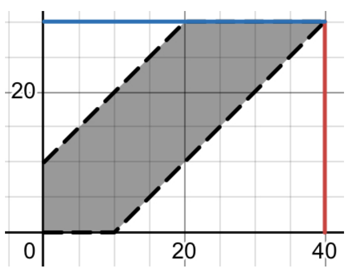
24. (Sigma Liu) Atticus and Ray plan to meet in a library. Normally, it will take Atticus 40 minutes to reach there and Ray 30 minutes. However, on each person's way there is extraordinarily congested traffic. It will consume Atticus m extra minutes and Ray n extra minutes, in which m is a random real number chosen uniformly from the interval $[0, 30]$ and n a random number chosen uniformly from the interval $[0, 40]$. If they leave their homes simultaneously, find the possibility of Ray waiting for Atticus, but for a duration of less than 20 minutes.

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{3}{8}$ (D) $\frac{11}{24}$ (E) $\frac{13}{24}$

Answer: $\frac{11}{24}$

Solution: Atticus arrives in $40 + m$ minutes, and Ray arrives in $30 + n$ minutes.

Since Ray waits for Atticus for less than 20 minutes, $30 + n < 40 + m < 30 + n + 20$, which means $n - 10 < m < n + 10$. We graph the relationship between n and m.



The shaded area is between the graphs of $m = n - 10$ and $m = n + 10$. Since m and n are both chosen uniformly from the interval $[0, 30]$ and $[0, 40]$, the wanted probability is area of shaded region over area of rectangle, which equals

$$\frac{30 \cdot 40 - \frac{1}{2} \cdot 30 \cdot 30 - \frac{1}{2} \cdot 20 \cdot 20}{30 \cdot 40} = \boxed{\text{(D) } \frac{11}{24}}.$$

25. (Mr. Li) In summer, Elmo stayed at Grover's house. Elmo found that Grover went to library to volunteer every 4 days and went swimming every 3 days. Elmo remembered that Grover went to library to volunteer on the first day of his arrival and went swimming on the last day of his stay. If Grover went to library to volunteer and went swimming on the same day four times during Elmo's stay, at most how many days Elmo stayed at Grover's house this summer?

(A) 37 (B) 49 (C) 54 (D) 66 (E) 71

Answer: 54

Solution: Suppose Elmo stayed at Grover's house for n days. The equations $x \equiv 1 \pmod 4$ and $x \equiv n \pmod 3$ has 4 solutions for $1 \leq x \leq n$.

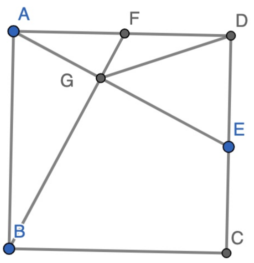
(a) $n \equiv 1 \pmod 3$: $x \equiv 1$ (mod 1)2. Grover goes volunteer and swimming on day 1, 13, 25, and 37. $37 \leq n \leq 49$ and $n \equiv 1 \pmod 3$, so $n = 46$.

(b) $n \equiv 2 \pmod 3$: $x \equiv 5 \pmod{12}$. Grover goes volunteer and swimming on day 5, 17, 29, and 41. $n = 50$.

(c) $n \equiv 0 \pmod 3$: Grover goes volunteer and swimming on day 9, 21, 33, and 45. $n = 54$.

Therefore, Elmo stays at Grover' house for at most $\boxed{\text{(C) } 54}$ days.

26. (Peter) As shown in the diagram, $ABCD$ is a square with side length 2. E is a point that moves freely on the segment CD, and F lies on AD such that BF is perpendicular to AE. What is the minimum possible length of DG?



(A) 1 (B) $\sqrt{5} - 1$ (C) $\sqrt{2}$ (D) $\frac{3}{2}$ (E) $\sqrt{3}$

Answer: $\sqrt{5} - 1$

Solution: Since BF is perpendicular to AE, $\angle AGB = 90°$. This means G lies on the semicircle with diameter AB.

Let M be midpoint of AB, then G lies on the circle centered at M with radius 1. In $\triangle DGM$, $DG + MG = DG + 1 \geq DM$. By Pythagorean Theorem, $DM = \sqrt{AD^2 + AM^2} = \sqrt{5}$. Therefore, $DG \leq \boxed{\text{(C) } \sqrt{5} - 1}$.

27. (Kevin Yang) Positive integers x, y, z, w satisfy $xy - zw = 1000$. Find the minimum possible value of $x + y + z + w$.

(A) 68 (B) 70 (C) 72 (D) 74 (E) 76

Answer: 70

Solution: Notice that $32^2 = 1024$. Moreover, we have $xy \le \frac{1}{4}(x+y)^2$, since $\frac{1}{4}(x+y)^2 - xy = \frac{1}{4}(x-y)^2 \ge 0$. If $x + y \le 31 + 32 = 63$, then $xy \le \frac{1}{4} \cdot 63^2 < 1000$. Since z and w are positive, $zw > 0$. And $xy > 1000$, so $x + y \ge 64$.

When $x + y = 64$, let $x = 32 + a, y = 32 - a$, then $xy = (32 + a)(32 - a) = 1024 - a^2 > 1000$, so $a^2 < 24$ and $a \le 4$. $zw = xy - 1000 = 24 - a^2$.

- $a = 0 : zw = 24 = 4 \cdot 6$. The least value of $x + y + z + w = 64 + 4 + 6 = 74$.
- $a = 1 : zw = 23 = 1 \cdot 23$. The least value of $x + y + z + w = 64 + 1 + 23 = 88$.
- $a = 2 : zw = 20 = 4 \cdot 5$. The least value of $x + y + z + w = 64 + 4 + 5 = 73$.
- $a = 3 : zw = 15 = 3 \cdot 5$. The least value of $x + y + z + w = 64 + 3 + 5 = 72$.
- $a = 4 : zw = 8 = 2 \cdot 4$. The least value of $x + y + z + w = 64 + 2 + 4 = 70$.

If $x + y + z + w < 70$, by the cases above, $x + y$ can't be 64, so $x + y \ge 65$ and $z + w \le 4$. $zw \le \frac{1}{4} \cdot 4^2 = 4$. Therefore, $xy \le 1004$. However, none of 1001 to 1004 can be expressed as a product of two numbers with sum less than 69, which means it's impossible.

Hence, the minimum possible value of $x + y + z + w$ is $\boxed{\text{(B) } 70}$.

28. (Mr. Li) PRISMS students hold PROM (Dance Party) every May. Last year in the PROM, every girl danced with 5 boys and 3 girls, while every boy danced with 4 girls and 4 boys. If 360 pairs of students danced at the party, how many girls attended the PROM?

(A) 20 (B) 25 (C) 40 (D) 50 (E) 60

Answer: 40

Solution: Let there be x girls and y boys. We have the following equations.

$$\begin{cases} 5x = 4y \\ 5x + 3x + 4y = 360. \end{cases}$$

By solving the equations, we get $(x, y) = (40, 50)$. Hence, there are $\boxed{\text{(C) } 40}$ girls.

29. (Sigma Liu) Eva wrote a number with two decimal digits in base-6: $(0.\overline{xy})_6$ on a blackboard. For example, if she wrote $(0.14)_6$, then the number equals $\frac{1}{6} + \frac{4}{6^2}$. Spencer converted it to a repeating decimal in base-7: $(0.\dot{z}\dot{w})_7$. Victor noticed that $x + y = z + w$, and he converted it to base-10. Find the sum of the different digits appearing in this number's base-10 representation. (If a number appears in the repeating unit, it is only added once)

(A) 11 (B) 14 (C) 16 (D) 17 (E) 20

Answer: 16

Solution: In base-6, $(0.\overline{xy})_6 = \frac{1}{6} \cdot x + \frac{1}{6^2} \cdot y$.

In base-7, $(0.\dot{z}\dot{w})_7 = \frac{1}{7} \cdot z + \frac{1}{7^2} \cdot w + \frac{1}{7^3} \cdot z + \frac{1}{7^4} \cdot w + \cdots = (\frac{1}{7} \cdot z + \frac{1}{7^2} \cdot w) \cdot (1 + \frac{1}{7^2} + \frac{1}{7^4} + \dots)$.

$1 + \frac{1}{7^2} + \frac{1}{7^4} + \cdots = \frac{1}{1-\frac{1}{7^2}} = \frac{49}{48}$. Therefore, $(0.\dot{z}\dot{w})_7 = \frac{49}{48}(\frac{1}{7} \cdot z + \frac{1}{7^2} \cdot w) = (0.\overline{xy})_6 = (\frac{1}{6} \cdot x + \frac{1}{6^2} \cdot y)$. Simplifying it gives $7z + w = 8x + \frac{4}{3}y$.

Since x, y, z, w are integers less than 7, $3x + \frac{4}{3}y$ is an integer. So $y = 0$ or 3.

(a) $y = 0$: $7z + w = 8x$, and $x = z + w$. Therefore, $7z + w = 8z + 8w$, contradiction to the fact that x, y, z, w are all positive.

(b) $y = 3$: $7z + w = 8x + 4$. Since $x + y = z + w$, we substitute $z + w = x + 3$ and get $6z+x+3 = 8x+4$, $7x = 6z-1$. Therefore, $(x, y, z, w) = (5, 3, 6, 2)$, and $\frac{5}{6}+\frac{3}{36} = \frac{11}{12} = 0.91\dot{6}$.

Sum of different digits is $9 + 1 + 6 = \boxed{\text{(C) } 16.}$

30. (Felicity) We know $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$. Find $\frac{1}{1^2 \cdot 2^2} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{3^2 \cdot 4^2} + \dots$.

(A) $\frac{\pi^2}{6} - \frac{3}{2}$ (B) $\frac{\pi^2}{3} - 3$ (C) $\frac{\pi^4}{36}$ (D) $\frac{\pi^4}{36} - \frac{\pi^2}{6}$ (E) $\frac{\pi^2}{12}$

Answer: $\frac{\pi^2}{3} - 3$

Solution:

$$\begin{aligned}
\frac{1}{1^2 \cdot 2^2} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{3^2 \cdot 4^2} + \dots &= \left(\frac{1}{1 \cdot 2}\right)^2 + \left(\frac{1}{2 \cdot 3}\right)^2 + \left(\frac{1}{3 \cdot 4}\right)^2 + \dots \\
&= \left(\frac{1}{1} - \frac{1}{2}\right)^2 + \left(\frac{1}{2} - \frac{1}{3}\right)^2 + \left(\frac{1}{3} - \frac{1}{4}\right)^2 + \dots \\
&= \frac{1}{1^2} - \frac{2}{1 \cdot 2} + \frac{1}{2^2} + \frac{1}{2^2} - \frac{2}{2 \cdot 3} + \frac{1}{3^2} + \frac{1}{3^2} - \frac{2}{3 \cdot 4} + \frac{1}{4^2} + \dots \\
&= 2 \cdot \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) - 1 - 2 \cdot \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots\right) \\
&= \frac{\pi^2}{3} - 1 - 2 \cdot \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots\right) \\
&= \boxed{\text{(D) } \frac{\pi^2}{3} - 3}.
\end{aligned}$$

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