PROM^2 Individual Solutions

Problem 1

What is the first year that is a prime number after 2026?

(A) 2027 **(B)** 2028 **(C)** 2029 **(D)** 2030 **(E)** 2031

Solution

$\boxed{\textbf{(A) } 2027}$ is prime and the smallest integer after 2026. Therefore, it is the smallest prime after 2026.

Problem 2

At the PRISMS Easter egg hunt, there are 100 children searching for eggs. 60 children found a red egg. 45 children found a blue egg. 25 children found both a red and a blue egg. How many children are unlucky and found none of the eggs?

(A) 10 **(B)** 20 **(C)** 30 **(D)** 40 **(E)** 50

Solution

The total number of children who found an egg is $60 + 45 - 25 = 80$ since each child who found two eggs is counted twice - once in the red egg group and once in the blue egg group. Therefore, $100 - 80 = \boxed{\textbf{(B) } 20}$ children did not find any eggs.

Problem 3

Dragonflies often carry out a behavior known as "obelisking," where they point their abdomen (tail/back) toward the sun, allowing them to cool off as less sun hits their body. If a dragonfly was initially at a temperature of 40 degrees Celsius, and as it obelisks it cools at an average rate of 2 degrees/minute, how many minutes would it take for it to cool to 31 degrees Celsius?

(A) 3.5 **(B)** 4 **(C)** 4.5 **(D)** 5 **(E)** 5.5

Solution

Let t be the number of minutes that pass. Then $31 = 40 - 2t$, so $t = \boxed{\textbf{(C) } 4.5}$

Problem 4

A rectangle has a perimeter of 60 and integer side lengths. What is the largest possible area?
$\textbf{(A) } 200 \qquad \textbf{(B) } 216 \qquad \textbf{(C) } 220 \qquad \textbf{(D) } 225 \qquad \textbf{(E) } 256$

Solution

Let the two side lengths be a and b. $a + b + a + b = 60$, so $a + b = 30$. Then the area of the rectangle is

$$ab = a(30 - a) = 30a - a^2 = 225 - (a - 15)^2.$$

This expression is then maximized when $(a - 15)^2$ is minimized, which is when $a = 15$, or when the rectangle is a square. This square has an area of $15 \cdot 15 = \boxed{\textbf{(D) } 225}$.

Problem 5

Pac-Man is located at the bottom-left square of a 4×4 grid of squares and needs to reach the top-right square to eat a cherry. If he can only move one square to the right or one square up at a time, how many different shortest paths are there?
$\textbf{(A) } 12 \qquad \textbf{(B) } 16 \qquad \textbf{(C) } 20 \qquad \textbf{(D) } 24 \qquad \textbf{(E) } 32$

Solution

Pac-Man must move 3 units right and 3 units up to get to the cherry. This is equivalent to the number of ways to rearrange $UUURRR$, where U symbolizes moving up and R symbolizes moving to the right, or $\binom{6}{3} = \frac{6!}{3!3!} = \boxed{\textbf{(C) } 20}$.

Problem 6

A 2-digit number is exactly 4 times the sum of its digits. How many 2-digit numbers satisfy the condition?
$\textbf{(A) } 3 \qquad \textbf{(B) } 4 \qquad \textbf{(C) } 5 \qquad \textbf{(D) } 6 \qquad \textbf{(E) } 7$

Solution

Let a be its tens digit, and let b be its ones digit. Then we can rewrite the given condition as $10a + b = 4(a + b)$, or $2a = b$. This is satisfied where $1 \leq a \leq 9$ and $0 \leq b \leq 9$ by $(a, b) = (1, 2), (2, 4), (3, 6), (4, 8)$, so there are $\boxed{\textbf{(B) } 4}$ such 2-digit numbers.

Problem 7

At the PROM^2 competition, participant IDs are three-digit positive integers. A student notices her ID leaves remainder 1 when divided by 12, 15, and 16. What is the smallest possible ID?

(A) 108 **(B)** 161 **(C)** 241 **(D)** 481 **(E)** 721

Solution

Let the ID be x. Then, $x-1$ is divisible by 12, 15, and 16, or $x-1 = \text{lcm}(12, 15, 16) \cdot c = 240c$ for some $c \in \mathbb{Z}$. The smallest value of x such that x is a 3 digit integer is achieved when $c = 1$, so $x = \boxed{\textbf{(C) } 241}$.

Problem 8

Michael can build an FTC robot in 42 days. Julianne can build an FTC robot in 56 days. If they work together, how many days will they take to build an FTC robot?

(A) 14 **(B)** 24 **(C)** 28 **(D)** 30 **(E)** 35

Solution

Michael builds $\frac{1}{42}$ FTC robots in 1 day, and Julianne builds $\frac{1}{56}$ FTC robots in 1 day, so their combined rate is $\frac{1}{42} + \frac{1}{56} = \frac{1}{24}$ FTC robots in 1 day. Therefore, it takes $\boxed{\textbf{(B) } 24}$ days for them to build one FTC robot together.

Problem 9

How many zeroes are at the end of the product $40^{50} \cdot 50^{40}$?

(A) 40 **(B)** 50 **(C)** 90 **(D)** 130 **(E)** 140

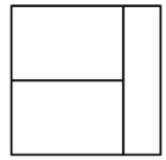
Solution

The given product can be rewritten as $4^{50} \cdot 5^{40} \cdot 10^{50} \cdot 10^{40} = 2^{60} \cdot 2^{40} \cdot 5^{40} \cdot 10^{90} = 2^{60} \cdot 10^{40} \cdot 10^{90} = 2^{60} \cdot 10^{130}$. 2^{60} does not have any trailing zeroes, so the product ends in $\boxed{\textbf{(D) } 130}$ zeroes.

Problem 10

A square with perimeter 168 is split into 3 rectangles with equal perimeter, as shown in the diagram. What is the perimeter of one such rectangle?

(A) 56 **(B)** 98 **(C)** 105 **(D)** 112 **(E)** 126



Solution

The square has side length $\frac{168}{4} = 42$, so we know that the shorter side length of the two left rectangles is 21. Let the longer side length be x. The the side lengths of the other rectangle are then 42 and $42 - x$. By the given condition, $21 + x = 42 - x + 42$, so $x = \frac{63}{2}$. Thus, the perimeter is $2 \times (21 + \frac{63}{2}) = \boxed{\textbf{(C)}\ 105}$.

Problem 11

How many different letter arrangements can be made by rearranging the letters in the word "AVATAR"?

(A) 60 **(B)** 120 **(C)** 240 **(D)** 360 **(E)** 720

Solution

The number of ways to rearrange the letters, assuming that they are all distinct, is 6!. However, in this way, the three As are counted as different, so each distinct arrangement is double counted 3! times. Thus, the total number of distinct rearrangements is $\frac{6!}{3!} = \boxed{\textbf{(B)}\ 120}$.

Problem 12

A sequence of five consecutive odd integers has an average value of 19. What is the product of the smallest and the largest integer in this sequence?

(A) 325 **(B)** 345 **(C)** 357 **(D)** 513 **(E)** 2026

Solution

Let the five consecutive odd integers be denoted as $a, a+2, a+4, a+6, a+8$. Then their sum is $a + a + 2 + a + 4 + a + 6 + a + 8 = 5a + 20$, so their average is $\frac{5a+20}{5} = a + 4 = 19$. The smallest integer in the sequence, or a, is 15, and the largest integer is $a + 8 = 23$. The product is then $\boxed{\textbf{(B)}\ 345}$.

Problem 13

In the Cartesian coordinate system, the line $y = 2x+2$ is reflected over the line $y = x$. What is the new y-intercept of the new line we obtained?

(A) -2 **(B)** -1 **(C)** 1 **(D)** 2 **(E)** 4

Solution

Reflecting $y = 2x + 2$ over $y = x$ is equivalent to switching y and x in $y = 2x + 2$, so the new equation is $x = 2y + 2$. The y-intercept can be calculated by substituting $x = 0$, so $y = \boxed{\textbf{(B)}\ -1}$.

Problem 14

The quadratic equation $x^2 - 9x + k = 0$ has two roots, one of which is exactly twice the value of the other. What is the value of the constant k?

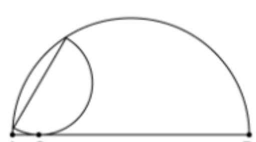
(A) 0 **(B)** 2 **(C)** 8 **(D)** 18 **(E)** 72

Solution

Let the two roots be r and $2r$. Then $(x - r)(x - 2r) = x^2 - 3rx + 2r^2 = x^2 - 9x + k$. $3r = 9$, so $r = 3$. Thus, $k = 2 \cdot r^2 = 2 \cdot 3^2 = \boxed{\textbf{(D)}\ 18}$.

Problem 15

A semicircle has vertices A and B. A smaller semicircle whose vertices lie on arc AB is tangent to segment AB at C. If $AC = 8$ and $BC = 36$, what is the radius of the smaller semicircle?



(A) 9 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 15

Solution

Let O_1 be the center of the larger semicircle, and let O_2 be the center of the smaller semicircle. Let the intersection of O_1O_2 with the smaller semicircle be Y, and let x be the length of O_1Y. We know that $\angle O_2CY = 90$, $O_2C = r$, $O_1C = 14$, so $(x+r)^2 = 14^2+r^2 \implies x^2+2r = 14^2$. Let us now use Power of a Point on the smaller semicircle. We get

$$r^2 = (22 - x - r)(22 + x + r) = 44^2 - (x + r)^2$$

and then

$$2r^2 = 44^2 - x^2 - (2r)x = 44^2 - 14^2,$$

so $r = \boxed{\textbf{(C) } 12}$.

Problem 16

How many permutations of "21067", when read as a base-ten number, are 5-digit integers that are divisible by 11?

(A) 8 **(B)** 12 **(C)** 16 **(D)** 24 **(E)** 31

Solution

A permutation $abcde$ is divisible by 11 iff $a + c + e \equiv b + d \pmod{11}$. We proceed by casework.

Case 1: b and d are 2 and 6 in some order. The case where $b = 2$ is identical to the case where $b = 6$, so we assume without loss of generality that $b = 2$ and count this one twice. One of c and e is 0, and the other is one of 1 and 7, so there are $2 \times 2 \times 2 = 8$ permutations in this case.

Case 2: b and d are 1 and 7 in some order. This case is identical to the above case, so there are 8 permutations in this case.

The total number of permutations is $\boxed{\textbf{(C) } 16}$.

Problem 17

If $x + y = 7$ and $x^3 + y^3 = 154$, find xy.

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Solution

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) = (x + y)((x + y)^2 - 3xy).$$

Substituting in the given values,

$$154 = 7(49 - 3xy),$$

so $xy = \boxed{\textbf{(B) } 9}$.

Problem 18

Given $(\text{PROM})^2 = 62916624$, where each letter represents a different digit, and that 661, a prime number, divides PROM, how many factors does PROM have?

(A) 6 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 30

Solution

Let $x = \text{PROM}$. If $661 \mid x^2$, $661 \mid x$. $3 \mid (6+2+9+1+6+6+2+4)$, so $3 \mid x$, and $8 \mid 624$, so $8 \mid x^2$ and $4 \mid x$. Then $x = c \cdot 661 \cdot 4 \cdot 3 = 7932c$ for some $c \in \mathbb{Z}$. If $c > 1$, x has 5 or more digits, so $\text{PROM} = 7932 = 2^2 \cdot 3 \cdot 661$, which has $(2+1)(1+1)(1+1) = \boxed{\textbf{(B) } 12}$.

Problem 19

Four secret agents each grab one briefcase in the dark. When the lights turn on, they realize that none of them grabbed their own original briefcase. In how many different ways could this situation occur?

(A) 1 **(B)** 3 **(C)** 6 **(D)** 9 **(E)** 23

Solution

Assume without loss of generality that Agent A grabs Agent B's briefcase. Then we must consider three cases.

The first case is that Agent B grabs Agent A's briefcase. Then Agent C has to have grabbed Agent D's briefcase, and Agent D has to have grabbed Agent C's briefcase. This case contributes 1 way.

The second case is that Agent B grabs Agent C's briefcase. Then Agent D must grab Agent A's briefcase, and Agent C must grab Agent D's. This case contributes 1 way.

The third case, where Agent B grabs Agent D's, is identical to the second case, so it also contributes 1 way, for a total of 3 ways.

However, this assumes that Agent A grabbed Agent B's briefcase, which is identical to the cases where Agent A grabs Agent C or D's briefcase, so there are $3 \cdot 3 = \boxed{\textbf{(D) } 9}$.

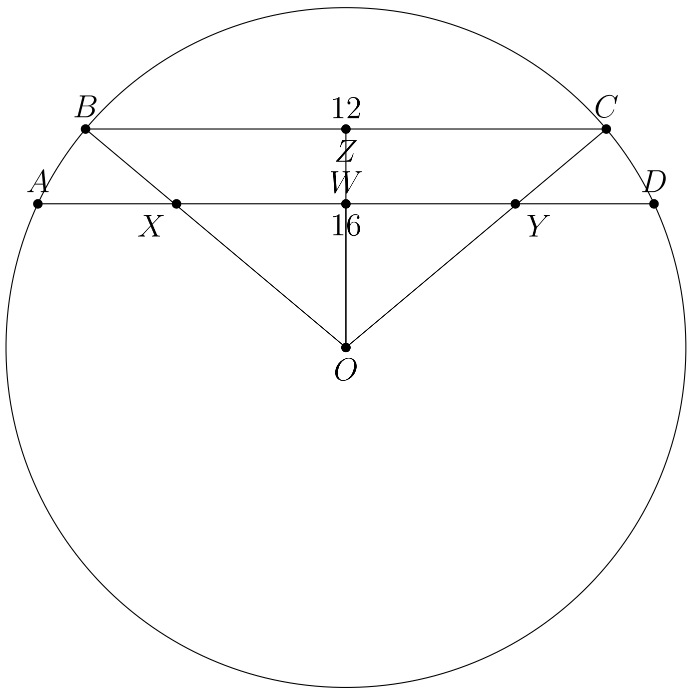
Remark. *This is also equivalent to just counting the number of derangements of 4 objects, so by applying the derangement formula, we get* $4! \cdot (\frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!}) = \boxed{\textbf{\textit{(D) 9}}}$.

Problem 20

Circle ω with center O has a radius of 10. Points A, B, C, and D lie on ω in that order such that $\overline{BC} \parallel \overline{AD}$, $\overline{BC} = 12$, and $\overline{AD} = 16$. Segments $\overline{OB}$ and $\overline{OC}$ intersect $\overline{AD}$ at X and Y, respectively. Find $\overline{XY}$.

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Solution



We know $\overline{BZ} = 6$, so $\overline{OZ} = 8$. We also know that $\overline{AW} = 8$, so $\overline{OW} = 6$, so $\frac{\overline{XY}}{\overline{OW}} = \frac{\overline{BC}}{\overline{OZ}}$, and consequently $\overline{XY} = \boxed{\textbf{(B) } 9}$.

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