PROM^2 Team Solutions

Problem 1

Define $n!$ as $n! = n \times (n-1) \times (n-2) \times ... \times 1$, where $0! = 1$. Find the value of $2!+0!+2!+6!$.

Solution

$2! + 0! + 2! + 6! = 2 + 1 + 2 + 720 = \boxed{725}$.

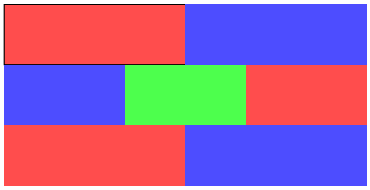
Problem 2

What is the minimum number of colors required to color the regions in the diagram so that no two adjacent regions share the same color? (Each region must be colored using exactly one color)

Solution

Assume for the sake of contradiction that 2 colors is sufficient. Then the first region in the top row could be colored red, and the second region in the top row could be colored blue. Then the first region in the second row would have to be blue, and the second region in the second row would have to be red, but it is also adjacent to the top left region, so 2 colors is insufficient.

Below we present a coloring with $\boxed{3}$ colors that works.



Problem 3

Anton is currently 5 times as old as his niece and 14 years younger than his brother. Given that his brother's age is 7 times the age of his niece, how old will Anton be in 15 years?

Solution

Let Anton's age be a, let his niece's age be n, and let his brother's age be b. We know that $a = 5n = b - 14$, and $b = 7n$, so $5n = 7n - 14$, and consequently, $n = 7$. Then $a = 35$, and $a + 15 = \boxed{50}$.

Problem 4

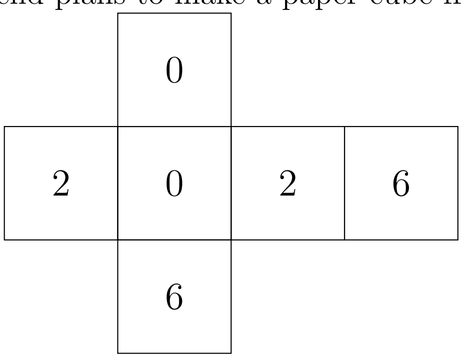
Arrange the three numbers $A = 2^{300}$, $B = 3^{200}$ and $C = 5^{100}$ in increasing order.

Solution

$A = 2^{300} = 8^{100}$, $B = 3^{200} = 9^{100}$, and $C = 5^{100}$, so $\boxed{C < A < B}$.

Problem 5

Euclid plans to make a paper cube from the following paper:



After constructing it, he multiplies the numbers on all pairs of faces that share an edge, and sums all of these products together. What is his final sum?

Solution

Solution 1 We compute the sum of the products of all the faces together without the condition that only adjacent faces are multiplied together. Then we have counted multiplying the same face together once and multiplying pairs of opposite faces together twice. Then we have the product of each pair counted twice, so we divide by 2.

$$\frac{(2+0+0+6+2+6)^2 - 2^2 - 0^2 - 0^2 - 6^2 - 2^2 - 6^2 - 2\cdot(2\cdot 2 + 0\cdot 6 + 0\cdot 6)}{2} = \boxed{84}$$

Solution 2 Opposite faces have the same set of adjacent pairs, so we sum the following way:

$(0+6)(2+0+2+6)+(2+2)(0+0+6+6)+(0+6)(2+2+0+6) = 6{\cdot}10+4{\cdot}12+6{\cdot}10 = 60+48+60 = 168$

However, we have counted each product twice, so the final answer is $\boxed{84}$.

Solution 3 Computing this directly is also not too difficult, as it is just 12 products. This also comes out to $\boxed{84}$.

Problem 6

Blorp is tasked to create a new calendar for his planet, Borp. Unfortunately, the number of days in a Borp year is very annoying. If he uses 7-day weeks like on Earth, then there'll be an extra 5 days. If he uses 16 instead, there'll be 9 extra days. Eventually, he settles with 17 days per week, since that results in a whole number of weeks. What is the minimum number of days in a Borp year?

Solution

Let the number of days be $17n$ for some $n \in \mathbb{Z}$. Then $17n \equiv 5 \pmod{7}$, and $17n \equiv 9 \pmod{16}$. Consequently, $n \equiv 4 \pmod{7}$, and $n \equiv 9 \pmod{16}$, so $n \equiv 25 \pmod{112}$, and $17n = \boxed{425}$.

Problem 7

If 2 divides the sum of 5 consecutive primes, what is the product of the primes?

Solution

We know that 2 needs to be the first prime because the sum is even; if all the primes were greater than or equal to 3, the sum would be odd. Therefore, the primes are 2, 3, 5, 7, and 11, so the product is $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = \boxed{2310}$.

Problem 8

ΔABC has a right angle at B, and the lengths of AB and BC are 6 and 8, respectively. Given that circle O is tangent to all three sides of the triangle, what is the radius of circle O?

Solution

The inradius $r = \frac{A}{s} = \frac{\frac{6\cdot 8}{2}}{\frac{6+8+10}{2}} = \boxed{2}$.

Problem 9

In 2026, on how many days is the sum of the digits of the month and day equal to 6?

Solution

We proceed by casework.

In months where the sum of the digits of the month equal 1 (January and October), there are 3 dates: the 5th, the 14th, and the 23rd, so this case contributes 6 dates.

In months where the sum of the digits of the month equals 2 (February and November), there are 3 dates: the 4th, the 13th, and the 22nd, so this case contributes 6 dates.

In months where the sum of the digits of the month equals 3 (March and December), there are 4 dates: the 3rd, the 12th, the 21st, and the 30th, so this case contributes 8 dates.

In April, there are 3 dates: 4/20, 4/2, and 4/11.

In May, there are 2 dates: 5/1 and 5/10.

Across all cases, there are $\boxed{25}$ dates.

Problem 10

Al-Khwarizmi is finding the average of 54, 86, 87, 90, 93, 94, 95, and 100. However, he accidentally adds one of the values twice. If he gets an average of 88, what values did he repeat?

Solution

The average is $\frac{54+86+87+90+93+94+95+100+x}{9} = 88$, where x is the repeated value, so $x = \boxed{93}$.

Problem 11

Evaluate the exact value of the following product:

$$(1 - \frac{1}{2^2}) \times (1 - \frac{1}{3^2}) \times (1 - \frac{1}{4^2}) \times ... \times (1 - \frac{1}{10^2})$$

Let the exact value of the product be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution

$$(1 - \frac{1}{2^2}) \times (1 - \frac{1}{3^2}) \times (1 - \frac{1}{4^2}) \times ... \times (1 - \frac{1}{10^2}) =$$

$$\frac{2^2 - 1}{2^2} \times \frac{3^2 - 1}{3^2} \times \frac{4^2 - 1}{4^2} \times ... \times \frac{10^2 - 1}{10^2} =$$

$$\frac{(2-1)(2+1)}{2^2} \times \frac{(3+1)(3-1)}{3^2} \times ... \times \frac{(10-1)(10+1)}{10^2} =$$
$$\frac{1 \times 11}{2 \times 10} = \frac{11}{20}$$

11+20=$\boxed{31}$.

Problem 12

A staircase has 10 steps. You can climb either 1 or 2 steps at a time. In how many distinct ways can you reach the top from the bottom?

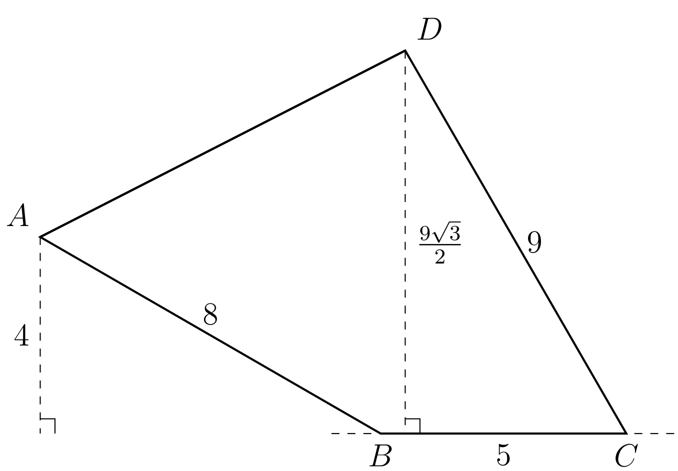
Solution

Let a_n be the number of ways to get to the nth step, where $a_n = a_{n-1} + a_{n-2}$ because we can reach step n either from step $n-1$ or from step $n-2$. From here we can either compute a_{10} directly or notice that $a_{10} = F_{11} = \boxed{89}$.

Problem 13

$A, B, C,$ and D are 4 points on a plane. Let $AB = 8, BC = 5, CD = 9$, $\angle ABC = 150, \angle BCD = 60$. What is the sum of all possible areas of triangle ΔABD?

Solution



We approach this problem using coordinate geometry. Let B be the origin $(0,0)$ and place C on the positive x-axis at $(5,0)$.

Since $\angle ABC = 150°$ and $AB = 8$, point A is located in the second quadrant (assuming it lies above the x-axis without loss of generality). Its coordinates are:

$$A = (8\cos 150°, 8\sin 150°) = (-4\sqrt{3}, 4)$$

Next, we determine the coordinates of D. We are given $CD = 9$ and $\angle BCD = 60^\circ$. Because the problem only specifies that the 4 points are on a plane without restricting the polygon's convexity, there are two possible valid configurations for D: it can lie on the same side of line BC as A, or on the opposite side.

Case 1: D is on the same side of line BC as A (as shown in the diagram).
The ray CD forms an angle of $180^\circ - 60^\circ = 120^\circ$ with the positive x-axis. Thus, the coordinates of D_1 are:

$$D_1 = (5 + 9\cos 120^\circ, 0 + 9\sin 120^\circ) = \left(5 - \frac{9}{2}, \frac{9\sqrt{3}}{2}\right) = \left(\frac{1}{2}, \frac{9\sqrt{3}}{2}\right)$$

Since B is the origin, the area of $\triangle ABD_1$ can be calculated directly using the coordinate area formula $\frac{1}{2}|x_A y_D - x_D y_A|$:

$$\text{Area}_1 = \frac{1}{2}\left|(-4\sqrt{3})\left(\frac{9\sqrt{3}}{2}\right) - (4)\left(\frac{1}{2}\right)\right| = \frac{1}{2}|-54-2| = 28$$

Case 2: D is on the opposite side of line BC from A.
The ray CD forms an angle of $180^\circ + 60^\circ = 240^\circ$ with the positive x-axis. Thus, the coordinates of D_2 are:

$$D_2 = (5 + 9\cos 240^\circ, 0 + 9\sin 240^\circ) = \left(\frac{1}{2}, -\frac{9\sqrt{3}}{2}\right)$$

The area of $\triangle ABD_2$ is:

$$\text{Area}_2 = \frac{1}{2}\left|(-4\sqrt{3})\left(-\frac{9\sqrt{3}}{2}\right) - (4)\left(\frac{1}{2}\right)\right| = \frac{1}{2}|54-2| = 26$$

Summing the areas of all possible configurations, we get $28 + 26 = \boxed{54}$.

Problem 14

What are the last 2 digits of 9 + 99 + ... + 99...9, where the last 99.9 is equal to $10^{99} - 1$?

Solution

We are finding this sum $\mod 100$. This can be rewritten as $9 + 99 + 99 + ... + 99 \equiv 9 + 99 \times 98 \equiv 9 + (-1) \times (-2) \equiv \boxed{11} \pmod{100}$.

Problem 15

There are 4 liars and 1 truth-teller in a classroom. When questioned by the teacher, they responded as follows:

Andy: Dave is the truth teller

Bella: I am the truth-teller

Charlotte: No, I'm the truth-teller!

Dave: Evan is a liar.

Evan: Charlotte is lying.

Who is the truth-teller?

Solution

Charlotte and Evan's statements contradict each other, so it must be one of them.

Assume for the sake of contradiction that Charlotte is the truth-teller. Then Evan must be a liar, but Dave also must be a liar, so Evan must be a truth-teller. This is a contradiction.

Then $\boxed{\text{Evan}}$ is the truth-teller, and all the statements are consistent.

Problem 16

The sum of the first 5 terms of a geometric sequence is 16. The sum of the next 5 terms is 24. What is the sum of the next 15 terms?

Solution

The sum of the first 5 terms is $a + ar + ar^2 + ar^3 + ar^4 = 16$, and the sum of the next 5 terms is $r^5(a + ar + ar^2 + ar^3 + ar^4) = 24$, so $r^5 = \frac{3}{2}$. The desired sum is $r^{10}(a + ar + ar^2 + ar^3 + ar^4)(1 + r^5 + r^{10}) = (\frac{3}{2})^2(16)(1 + (\frac{3}{2}) + (\frac{3}{2})^2) = \boxed{171}$.

Problem 17

Let $ABCD$ be a rectangle, and $EFGH$ be a square. If $AB = 6, AE = CG = 7$, and $AD = 8$, what is the greatest possible area of square $EFGH$?

Solution

Place E and G on line AC such that A and C are between E and G and E is closer to G. Then $EG = 24$, so the side length of $EFGH$ is $12\sqrt{2}$, and the area is $(12\sqrt{2})^2 = \boxed{288}$.

Problem 18

How many 3-digit numbers satisfy the following two conditions: its hundreds, tens, and units digits are three distinct non-zero even numbers, and the product of these three digits is a perfect square?

Solution

There are 4 possible combinations of 3 digits for the hundreds, tens, and units places: $\{2,4,6\}, \{2,4,8\}, \{2,6,8\}, \{4,6,8\}$. The only option among these for which all the digits multiply to a perfect square is $\{2,4,8\}$, and there are $3! = \boxed{6}$ ways to order them.

Problem 19

Find the sum of all distinct products $m \times n$ such that m and n are integers satisfying the equation $mn - 2m - 3n = 5$.

Solution

We use Simon's Favorite Factoring Trick to get $mn - 2m - 3n + 6 = (m-3)(n-2) = 11$. 11 is prime, so $(m-3, n-2) = (11,1), (1,11), (-11,-1), (-1,-11)$, and consequently $(m,n) = (14,3), (4,13), (-8,1), (2,-9)$. The sum of all $m \times n$ is $14 \times 3 + 4 \times 13 - 8 \times 1 - 9 \times 2 = \boxed{68}$.

Problem 20

A cat wants to drink milk. It is currently at $(0,41)$, and the water bowl is at $(6,7)$, but it also wants to pick up a snack on the line $y = 0$, so it must make a stop at some point on $y = 0$ on its way to the bowl. Let the minimum distance it must cover be $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime. Find $a + b$.

Solution

Reflect $(6,7)$ across $y = 0$. Then it is at $(6,-7)$, and the same distance is covered. This is equivalent to finding the minimum distance for some path from $(0,41)$ to $(6,-7)$. This is just the length of a straight line, or $\sqrt{6^2 + (41+7)^2} = 6\sqrt{65}$, so $a + b = \boxed{71}$.

Special thanks to Sierra Zhang '29 for writing the solutions.